3.7a Rewrite in polar & compute

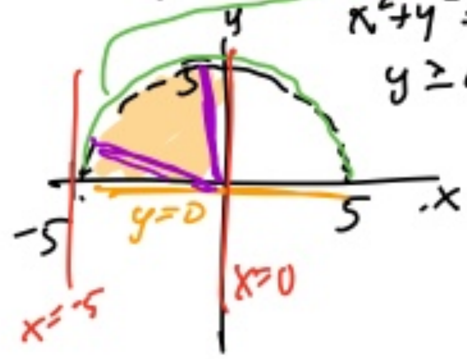
$$\int_{-5}^{0}\int_{0}^{\sqrt{25-x^2}} x^2\,dy\,dx$$

$-5 \le x \le 0$

$y=0 \to y=\sqrt{25-x^2}$

$x^2+y^2=25$

$y \ge 0$



$$\int_{\theta=\frac{\pi}{2}}^{\pi}\int_{r=0}^{5} (r\cos\theta)^2\, r\,dr\,d\theta$$

polar $0 \le r \le 5$

$\frac{\pi}{2} \le \theta \le \pi$

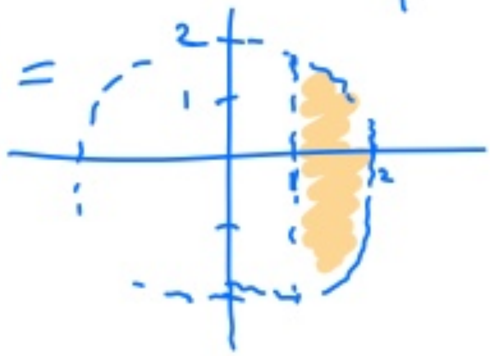
$$= \int_{\theta=\pi/2}^{\pi}\int_{r=0}^{5} r^3\cos^2\theta\,dr\,d\theta$$

$\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos(2\theta)$

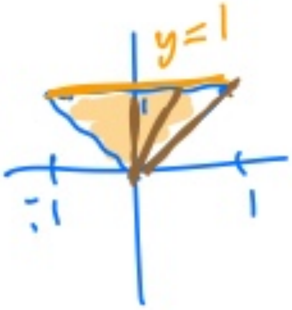
...

Find $\iint_R x\,dxdy$ and $\iint_R y\,dxdy$ in polar coordinates if (a) $R =$

(b) $R =$



(a) $R =$



$\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

$0 \leq r \leq \frac{1}{\sin\theta}$

$r\sin\theta = 1 \Rightarrow r = \frac{1}{\sin\theta} = \csc\theta$

integral $\iint_R x\,dy\,dx = \int_{\theta=\frac{\pi}{4}}^{3\pi/4} \int_{r=0}^{1/\sin\theta} (r\cos\theta)\, r\,dr\,d\theta$

$= \int_{\pi/4}^{3\pi/4} \int_{r=0}^{\frac{1}{\sin\theta}} r^2 \cos\theta\,dr\,d\theta = 0$ by symmetry (x on right is + x on left is same but −)

$\iint_R y\,dy\,dx = \int_{\theta=\frac{\pi}{4}}^{3\pi/4} \int_{r=0}^{\frac{1}{\sin\theta}} \underbrace{(r\sin\theta)\, r}_{r^2\sin\theta}\,dr\,d\theta$

$= \int_{\pi/4}^{3\pi/4} \left(\frac{r^3}{3}\Big|_0^{\frac{1}{\sin\theta}}\right) \sin\theta\,d\theta = \int_{\pi/4}^{3\pi/4} \frac{1}{3\sin^3\theta} \cdot \sin\theta\,d\theta$

$$= \int_{\pi/4}^{3\pi/4} \frac{1}{3}\csc^2\theta\, d\theta = -\frac{1}{3}\cot\theta \Big|_{\pi/4}^{3\pi/4}$$

$$= -\frac{1}{3}\left(\underbrace{\cot\left(\frac{3\pi}{4}\right)}_{-1} - \underbrace{\cot\left(\frac{\pi}{4}\right)}_{1}\right)$$

$$= -\frac{1}{3}(-2) = \boxed{\frac{2}{3}}.$$

Example Let V be the region between $z = x^2+y^2+1$ and $z=5$ in $\mathbb{R}^3$.

Find $\iiint_V (2x+3)\, dx\, dy\, dz$.

$z=5$ - plane at height $z=5$ (x & y can be anything)

$z = x^2+y^2+1$

$z=0 = x^2+y^2+1$ nothing

$z=1 = x^2+y^2+1 \Rightarrow x^2+y^2=0$

pt. $(0,0,1)$

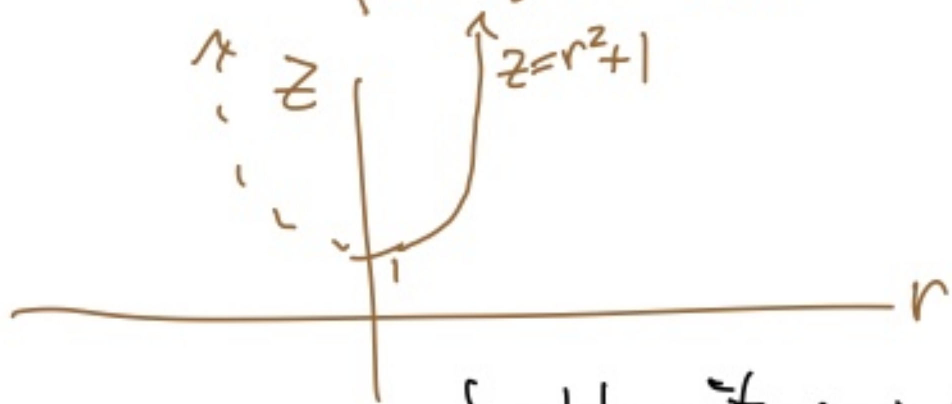
$z=2 = x^2+y^2+1 \Rightarrow x^2+y^2=1$

$z=3 = x^2+y^2+1 \Rightarrow x^2+y^2=2$

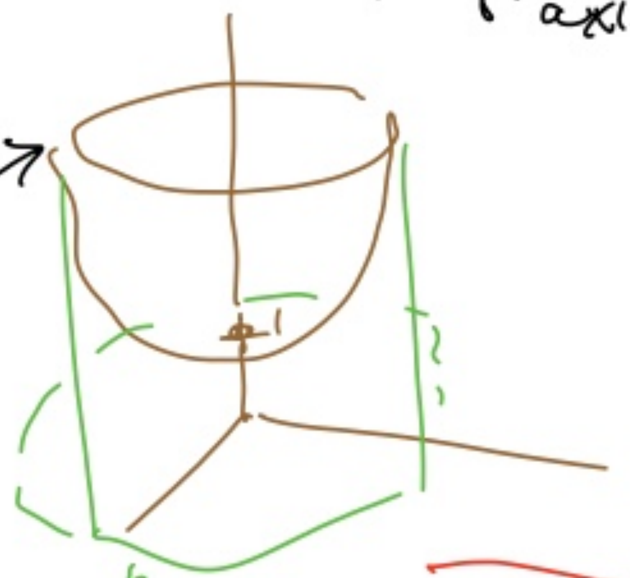
Notice our surface on the bottom is $z = x^2 + y^2 + 1$

polar (cylindrical) coordinates: $z = r^2 + 1$

$y = x^2 + 1$



whole picture → revolve the r axis around the z axis.



$z = 5$ and $z = x^2 + y^2 + 1$

$\cancel{5 \le z \le x^2 + y^2 + 1}$

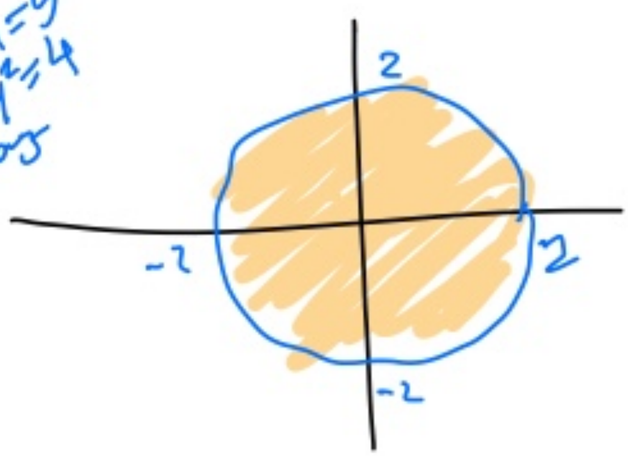
$\underset{r^2+1}{x^2+y^2+1} \le z \le 5$

What are the xy bounds? i.e. region in xy plane.

$x^2 + y^2 + 1 = 5$

$x^2 + y^2 = 4$

boundary



Polar:

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

<u>Our function:</u> $2x+3 = 2r\cos\theta+3$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2+1}^{5}(2r\cos\theta+3)\,dz\; r.dr.d\theta$$

$$= \int_{0}^{2\pi}\int_{r=0}^{2}(2r\cos\theta+3)z\Big|_{z=r^2+1}^{5}\; r\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{r=0}^{2}(2r\cos\theta+3)\left(5-(r^2+1)\right)r\,dr\,d\theta$$

$$5-r^2-1$$

$$(4-r^2)$$

$$\left[2r\cos\theta(4-r^2)+3(4-r^2)\right]r$$

$$\left[8r\cos\theta-2r^3\cos\theta+12-3r^2\right]r$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left(8r^2\cos\theta-2r^4\cos\theta+12r-3r^3\right)dr\,d\theta$$

$$\underset{\text{Fubinate}}{=}\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\left[(8r^2-2r^4)\cos\theta+12r-3r^3\right]d\theta\,dr$$

$$= \int_{r=0}^{2}(8r^2-2r^4)\sin\theta+(12r-3r^3)\theta\Big|_{0}^{2\pi}dr$$

$$= \int_{r=0}^{2} (12r - 3r^3)\, 2\pi \, dr$$

$$= 2\pi\left(6r^2 - \frac{3}{4}r^4\right)\Big|_0^2 = \left(6\cdot 4 - \frac{3}{\cancel{4}}\cdot \overset{4}{\cancel{16}}\right)2\pi$$

$$= (24 - 12)2\pi = \boxed{24\pi}$$

Note: $\iiint_{\text{region}} (2x+3)\,dx\,dy\,dz = \iiint 3\,dx\,dy\,dz$

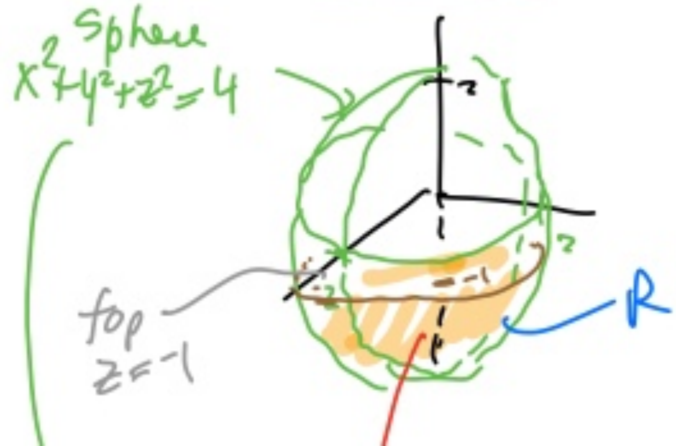
$2x$ integrates to 0
(+ & − parts cancel)

$$= 3\iiint 1\,dx\,dy\,dz$$

$= 3 \cdot$ volume of paraboloid piece.

$\Rightarrow$ volume $= \frac{1}{3}(24\pi) = 8\pi$

Example Find the volume of the portion of the ball of radius 2 centered at the origin that has $z \le -1$.



$$\text{Volume} = \iiint_R 1\,dx\,dy\,dz$$

$$\int_{z=\text{lower}}^{z=\text{upper}} dz$$

$-\sqrt{4-x^2-y^2} \le z \le -1$

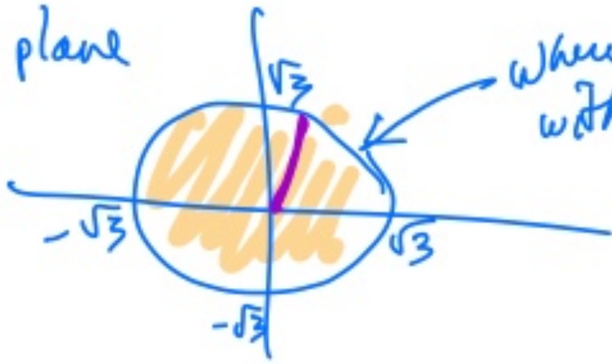
bottom

$z = -\sqrt{4-x^2-y^2}$

polar

$-\sqrt{4-r^2} \le z \le -1$

xy plane



when $z=-1$ intersects with $z = -\sqrt{4-x^2-y^2}$

$$\boxed{\begin{aligned} -1 &= -\sqrt{4-x^2-y^2} \\ 1 &= 4-x^2-y^2 \\ x^2+y^2 &= 3 \end{aligned}}$$

radius $\sqrt{3}$.

$0 \le r \le \sqrt{3}$

$0 \le \theta \le 2\pi$

$$\text{Volume} = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{3}} \int_{z=-\sqrt{4-r^2}}^{-1} dz \; r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{3}} z \Big|_{z=-\sqrt{4-r^2}}^{-1} \cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{3}} \left(-1+\sqrt{4-r^2}\right) r\,dr\,d\theta$$

$$\left(-r + r\sqrt{4-r^2}\right) dr$$

$-\frac{r^2}{2}$

$\int r\sqrt{4-r^2}\,dr$

$u = 4-r^2 \quad du = -2r\,dr$

$-\frac{1}{2}du = r\,dr$

$$= \int -\frac{1}{2}\sqrt{u}\,du$$

$$= \int -\frac{1}{2} u^{1/2}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$= -\frac{1}{3}u^{3/2} + C$$

$$= -\frac{1}{3}(4-r^2)^{3/2} + C$$

$$= \int_{\theta=0}^{2\pi} \left(-\frac{r^2}{2} + \frac{-1}{3}(4-r^2)^{3/2} \right)\Bigg|_{r=0}^{r=\sqrt{3}} d\theta$$

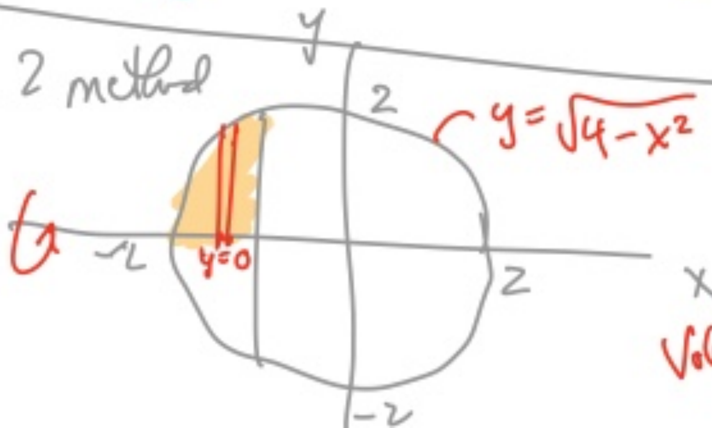
$$= \int_0^{2\pi} -\frac{(\sqrt{3})^2}{2} + \frac{-1}{3}\big(\underbrace{4-(\sqrt{3})^2}_{1}\big)^{3/2} - \left[0 + \frac{-1}{3}(4)^{3/2}\right] d\theta$$

$$= \int_0^{2\pi} \left(-\frac{3}{2} - \frac{1}{3} + \frac{1}{3}(8) \right) d\theta$$

$$-\frac{3}{2} + \frac{7}{3} = \frac{-9}{6} + \frac{14}{6} = \frac{5}{6}$$

$$= \int_0^{2\pi} \frac{5}{6}\,d\theta = \frac{5}{6}\cdot 2\pi = \boxed{\frac{5\pi}{3}}.$$

Calc 2 method



slice $\pi R_{dx}^2 = \pi\left(\sqrt{4-x^2}\right)^2 dx$
$= \pi(4-x^2)\,dx$

Volume $= \int_{-2}^{-1} \pi(4-x^2)\,dx$

$$= \pi\left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{-1}$$

$$= \pi\left(4(-1) - \frac{(-1)^3}{3}\right) - \pi\left(4(-2) - \frac{(-2)^3}{3}\right)$$

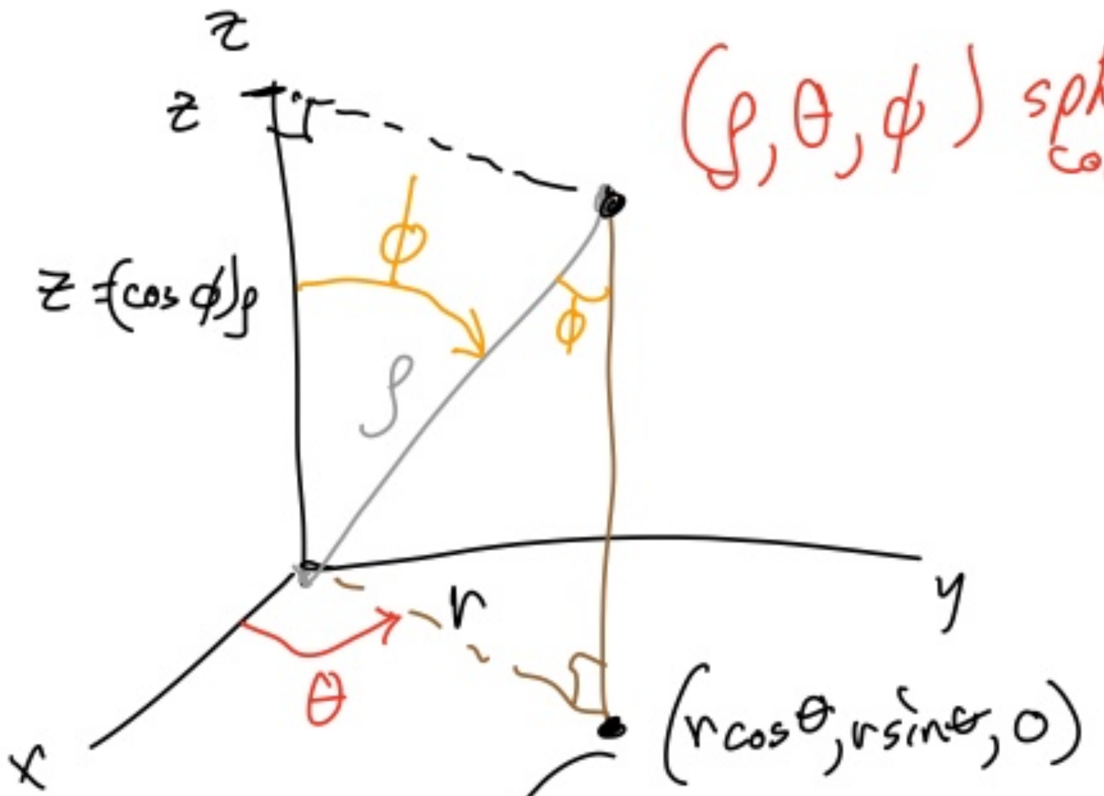
$$= \pi\left(-4 + \frac{1}{3}\right) - \pi\left(-8 + \frac{8}{3}\right)$$

$$= -4\pi + \frac{\pi}{3} + 8\pi - \frac{8\pi}{3}$$

$$= 4\pi - \frac{7\pi}{3} = \frac{(12-7)\pi}{3} = \boxed{\frac{5\pi}{3}}.$$

Spherical Coordinates

(ρ, θ, ϕ) spherical coordinates



$r = \rho \cdot \sin\phi$

$= (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, 0)$

Spherical Coordinates.

$$(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$